# Holonomy groups of flat manifolds with $R_\infty$ property

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**Abstract**

Let $M$ be a flat manifold. We say that $M$ has $R_\infty$ property if the Reidemeister number $R(f) = \infty$ for every homeomorphism $f\colon M \to M$. In this paper, we investigate a relation between the holonomy representation $\rho$ of a flat manifold $M$ and the $R_\infty$ property. In case when the holonomy group of $M$ is solvable we show that, if $\rho$ has a unique $\mathbb{R}$-irreducible subrepresentation of odd degree, then $M$ has $R_\infty$ property. The result is related to conjecture 4.8 from [3].

## 1 Introduction

Let $M^n$ be a closed Riemannian manifold of dimension $n$. We shall call $M^n$ flat if, at any point, the sectional curvature is equal to zero. Equivalently, $M^n$ is isometric to the orbit space $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete, torsion-free and co-compact subgroup of $O(n) \ltimes \mathbb{R}^n = \mathrm{Isom}(\mathbb{R}^n)$. From the Bieberbach theorem (see [1], [9], [10]) $\Gamma$ defines a short exact sequence of groups

(1.1) $$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

where $G$ is a finite group. $\Gamma$ is called a Bieberbach group and $G$ its holonomy group.

Since $\Gamma = \pi_1(M^n)$, any continuous map $f : M^n \to M^n$ induces a morphism $f_\sharp : \Gamma \to \Gamma$. We say that two elements $\alpha, \beta \in \Gamma$ are $f_\sharp$-conjugated if there exists $\gamma \in \Gamma$ such that $\beta = \gamma\alpha f_\sharp(\gamma)^{-1}$. The $f_\sharp$-conjugacy class $\{\gamma\alpha f_\sharp(\gamma)^{-1} \mid$

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$\gamma \in \Gamma\}$ of $\alpha$ is called a Reidemeister class of $f$. The number of Reidemeister classes is called the Reidemeister number $R(f)$ of $f$. A manifold $M^n$ has the $R_\infty$ property if $R(f) = \infty$ for every homeomorphism $f : M^n \to M^n$, see [3]. It is evident that we can also define the above number $R(f)$ for a countable discrete group $E$ and its automorphism $f$. We say that a group $E$ has $R_\infty$ property if $R(f) = \infty$ for any automorphism $f$. Moreover, the following groups (see [4] for the complete bibligraphy) have the $R_\infty$ property: non-elemtary Gromov-hyperbolic groups, Baumslag-Solitar groups $BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ except for $BS(1,1)$.

In this paper we shall consider the case of Bieberbach groups. We can define a holonomy representation $\rho : G \to \mathrm{GL}(n,\mathbb{Z})$ by the formula:

$$\forall g \in G, \rho(g)(e_i) = \tilde{g} e_i (\tilde{g})^{-1}, \tag{1.2}$$

where $e_i \in \Gamma$ are generators of the free abelian group $\mathbb{Z}^n$ for $i = 1, 2, ..., n$, and $\tilde{g} \in \Gamma$ such that $p(\tilde{g}) = g$. In this article we describe relations between $R_\infty$ property on the flat manifold $M^n$ (Bieberbach group $\Gamma$) and a structure of its holonomy representation. The connections between geometric properties of $M^n$ and algebraic properties of $\rho$ was already considered in different cases. For example, $\mathrm{Out}(\Gamma)$ is finite if and only if a holonomy representation is $\mathbb{Q}$-mutiplicity free and any $\mathbb{Q}$-irreducible components of a holonomy representaion is $\mathbb{R}$-irreducible, see [8]. A similar equivalence says that an Anosov diffeomorphism $f : M^n \to M^n$ exists if and only if any $\mathbb{Q}$-irreducible component of a holonomy representation that occurs with multiplicity one is reducible over $\mathbb{R}$, see [5]. We want to define conditions of this kind for the holonomy representation of a flat manifold with $R_\infty$ property. We already know that, in this way, the complete characteristic is not possible. There are examples [3, Th.5.9] of flat manifolds $M_1, M_2$ with the same holonomy representation such that $M_1$ has $R_\infty$ property and $M_2$ has not. In [3, Corollary 4.4] it is proved that if there exists na Anosov diffeomorphism $f : M^n \to M^n$ then $R(f)$ is finite and $M^n$ does not have the $R_\infty$ property. Moreover there exists $M$, such that its holonomy representation has $\mathbb{Q}$-irreducible component which is irreducible over $\mathbb{R}$ and occurs with multiplicity one, andt $M$ does not have the $R_\infty$ property, [3, Example 4.6]. Nevertheless in [3, Th. 4.7] is proved:

**Theorem 1.1.** *([3, Th. 4.7]) Let $M$ be a flat manifold with a holonomy representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ and let $\rho' : G \to \mathrm{GL}(n',\mathbb{Z})$ be a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation of $\rho$ such that $\rho'(G)$ is not $\mathbb{Q}$-conjugated to $\tilde{\rho}(G)$ for any other $\mathbb{Q}$-subrepresentation $\tilde{\rho}$ of $\rho$. Suppose moreover that for every $D' \in N_{\mathrm{GL}(n',\mathbb{Z})}(\rho'(G))$, there exists $A \in G$ such that $\rho'(A)D'$ has eigenvalue $1$. Then $M$ has the $R_\infty$ property.*

*Remark* 1.1. If we assume that

$$N_{\mathrm{GL}(n',\mathbb{Q})}(\rho'(G))/C_{\mathrm{GL}(n',\mathbb{Q})}(\rho'(G)) \cong Aut(G), \tag{1.3}$$

then the above requirement that $\rho'(G)$ is not $\mathbb{Q}$-conjugated to $\tilde{\rho}(G)$ is equivalent to the condition that $\rho' \subset \rho$ has multiplicity one. For example, if we take the diagonal representation $\rho : (\mathbb{Z}_2)^{2n} \to \mathrm{SL}(2n+1, \mathbb{Z})$ of the elementary abelian 2-group, then the above equation (1.3) is not satisfied for any $\mathbb{Q}$-irreducible subrepresentation of $\rho$.

We shall prove:

**Theorem 1.2.** *Let $M$ be a flat manifold with a holonomy representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{Z})$ and let $G$ be a solvable group and $\rho' : G \to \mathrm{GL}(n', \mathbb{Z})$ be a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation of $\rho$ of odd dimension. If $\rho'(G)$ is not $\mathbb{Q}$-conjugated to $\tilde{\rho}(G)$, for any other $\mathbb{Q}$-subrepresentation $\tilde{\rho}$ of $\rho$ then $M$ has the $R_\infty$ property.*

If we restrict our consideration to the class of finite groups which satisfy a condition (1.3) we have.

**Theorem 1.3.** *Let $M$ be a flat manifold with a holonomy representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{Z})$ and let $G$ be a solvable group and $\rho' : G \to \mathrm{GL}(n', \mathbb{Z})$ be a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation of $\rho$ of mulitiplicity one and odd dimension which satisfies a condition (1.3), then $M$ has the $R_\infty$ property.*

The above result is a corollary from [7, Th. 5.4.4], the Theorem 1.1 and the following theorem:

**Theorem A** *Let $G$ be a finite group with a non-trivial normal abelian subgroup $A$ and let $\rho\colon G \to \mathrm{GL}(n, \mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose $n$ is odd. Then for every $D \in N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue* 1.

*Remark* 1.2. A conjecture 4.8 in [3] says that the above theorem A is true for any finite group. We do not know whether it holds in general.

We prove **Theorem A** in the next section.

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## 2 Proof of Theorem A

**Theorem 2.1.** *Let $G$ be a finite group and $n$ be an odd integer. Let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a faithful representation of $G$, which is irreducible over $\mathbb{R}$. Then $\rho$ is irreducible over $\mathbb{C}$.*

*Proof.* Assume, that $\rho$ is reducible over $\mathbb{C}$ and let $\tau$ be any $\mathbb{C}$-irreducible subrepresentation of $\rho$. By [6, Theorem 2], the representation $\rho$ is uniquely determined by $\tau$ and, if $\chi$ is the character of $\tau$, then the character of $\rho$ is given by

$$\chi + \overline{\chi}.$$

Hence $\rho$ is of even degree. This proves the theorem. $\square$

For the rest of this section we assume, that $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ is an absolutely irreducible representation of $G$, where $n$ is an odd integer.

**Proposition 2.2.** *If $A$ is normal abelian subgroup of $G$, then $A$ is elementary abelian 2-group.*

*Proof.* Let $\tau$ be a $\mathbb{R}$-irreducible subrepresentation of $\rho_{|A}$. By Clifford's theorem ([2, Theorem 49.2]) all $\mathbb{R}$-subrepresentations of $\rho_{|A}$ are conjugates of $R$-irreducible subrepresentation $\tau$, i.e. there exists $g_1 = 1, g_2, \ldots, g_l \in G$ such that

$$\rho_{|A} = \tau^{(g_1)} \oplus \ldots \oplus \tau^{(g_l)}, \tag{2.1}$$

where

$$\forall_{1\leq i\leq l}\forall_{g\in G}\ \ \tau^{(g_i)}(g) = \tau(g_i^{-1}gg_i).$$

Let $a \in A$ be an element of order greater than 2. Since $\rho$ is faithful, there exists $1 \leq i \leq l$, such that $\tau^{(g_i)}(a)$ is a real matrix of order at least 2. Hence $\deg(\tau^{(g_i)}) = \deg(\tau) = 2$ and $n = \deg(\rho) = \deg(\rho_{|A}) = l\deg(\tau) = 2l$ is an even integer. This contradiction finishes the proof. $\square$

Since $A$ is an elementary abelian 2-group, the decomposition (2.1) may be realized over the rationals. By [2, Theorem 49.7] we may assume, that

$$\rho_{|A} = e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)}, \tag{2.2}$$

i.e. one-dimensional representations $\tau^{(g_1)}, \ldots, \tau^{(g_k)}$ occur with the same multiplicity $e = n/k$. Let $\rho_i := e\tau^{(g_i)}$, for $i = 1, \ldots, k$. By the suitable choice of basis of $\mathbb{Q}^n$ we may assume, that for every $a \in A, \rho(a)$ is a diagonal matrix, such that

$$\forall_{1\leq i\leq k}\mathrm{Img}(\rho_k) = \langle -\mathrm{I}\rangle, \tag{2.3}$$

where I is the identity matrix of degree $e$.

Since $A \lhd G$ and $\rho$ is faithful, we have

$$\rho(A) \lhd \rho(G) \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{m \in \mathrm{GL}(n,\mathbb{Q}) \mid m^{-1}\rho(A)m = \rho(A)\}.$$

In the next two subsections we will focus on the above normalizer.

## 2.1 Centralizer

In the beginning we describe the centralizer

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{m \in \mathrm{GL}(n,\mathbb{Q}) \mid \forall_{a \in A} m\rho(a) = \rho(a)m\}.$$

Let $m = (m_{ij}) \in \mathrm{GL}(n,\mathbb{Q})$ be a block matrix, such that $m\rho_{|A} = \rho_{|A}m$. We get

$$\begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix} \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k \end{pmatrix} = \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k \end{pmatrix} \begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix},$$

and thus

$$\forall_{1 \le i,j \le k} m_{ij}\rho_j = \rho_i m_{ij}.$$

Since for $i \neq j$, $\rho_i$ and $\rho_j$ have no common subrepresentation, by Schur's Lemma (see [2, (27.3)]) $m_{ij} = 0$ for $i \neq j$ and $m_{ii} \in \mathrm{GL}(n/k,\mathbb{Q})$, for $i = 1,\dots,k$. We just have proved

**Lemma 2.3.** *Let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Q})$ be a faithful, absolutely irreducible representation of finite group $G$ of odd degree $n$. Let $A$ be normal abelian subgroup of $G$, such that conditions* (2.2) *and* (2.3) *hold. Then*

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{\mathrm{diag}(c_1,\dots,c_k) \mid c_i \in \mathrm{GL}(n/k,\mathbb{Q}), i = 1,\dots,k\},$$

*where $k$ is equal to the number of pairwise nonisomorphic irreducible subrepresentations of $\rho_{|A}$.*

## 2.2 Normalizer

Since the group $A$ is finite, $\mathrm{Aut}(A)$ is a finite group. Moreover, we have a monomorphism

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))/C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \hookrightarrow \mathrm{Aut}(A).$$

Hence any coset $mC_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)), m \in N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ corresponds to some automorphism of $A$.

Let $\varphi \in \mathrm{Aut}(A)$ and $m = (m_{ij}) \in \mathrm{GL}(n, \mathbb{Q})$ be a block matrix, which represents this automorphism, with blocks of degree $n/k$, i.e.

$$\forall_{c \in C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))} \forall_{a \in A} (mc)\rho(a)(mc)^{-1} = m\rho(a)m^{-1} = \rho(\varphi(a)).$$

We have

$$\begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix} \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k \end{pmatrix} = \begin{pmatrix} \rho_1\varphi & & 0 \\ & \ddots & \\ 0 & & \rho_k\varphi \end{pmatrix} \begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix}.$$

Note, that

$$\forall_{1 \leq i \leq k} \mathrm{Img}(\rho_i) = \mathrm{Img}(\rho_i\varphi) = \langle -\mathrm{I} \rangle. \tag{2.4}$$

Since, for $i \neq j$, $\rho_i$ and $\rho_j$ does not have common subrepresentations, the same applies to $\rho_i\varphi$ and $\rho_j\varphi$. Hence, using Shur's lemma again for every $1 \leq i \leq k$ there exists exactly one $1 \leq j \leq k$, such that

$$m_{ji}\rho_i = \rho_j\varphi m_{ji}$$

and $m_{ji} \neq 0$. Moreover, $\det(m) \neq 0$ and also $\det(m_{ij}) \neq 0$. By (2.4) $\rho_i = \rho_j\varphi$ and there exists a permutation $\sigma \in S_k$, s.t.

$$m\mathrm{diag}(\rho_1, \dots, \rho_k)m^{-1} = \mathrm{diag}(\rho_{\sigma(1)}, \dots, \rho_{\sigma(k)}). \tag{2.5}$$

Let $\tau \in S_k$ be any permutation and let $P_\tau \in \mathrm{GL}(n, \mathbb{Q})$ be a block matrix, with blocks of degree $n/k$, such that

$$(P_\tau)_{i,j} = \begin{cases} \mathrm{I} & \text{if } \tau(i) = j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

where $1 \leq i, j \leq k$. By (2.5) we may take

$$m = P_\sigma$$

as a representative of a coset in $N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)/C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$, which realizes the automorphism $\varphi$.

Let

$$S := \{\tau \in S_k \mid P_\tau \in N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))\}.$$

Then $S$ is a subgroup of $S_k$ and

$$P := \{P_\tau \mid \tau \in S\}$$

is a subgroup of the normalizer. By the above and the Lemma 2.3, we get

**Proposition 2.4.** *The normalizer $N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ is a semidirect product of $C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ and $P$. Moreover*

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \cdot P \cong C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \rtimes S \cong \mathrm{GL}(n/k, \mathbb{Q}) \wr S,$$

*where $\mathrm{GL}(n/k, \mathbb{Q}) \wr S$ denotes the wreath product of $\mathrm{GL}(n/k, \mathbb{Q})$ and $S$.*

## 2.3 Properties of a group $G$

Let

$$C := \rho^{-1}\big(\rho(G) \cap C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))\big)$$

and

$$Q := \rho^{-1}(\rho(G) \cap P).$$

Then

$$G = C \cdot Q \cong C \rtimes Q \tag{2.7}$$

is a semidirect product of $C$ and $Q$. Without lose of generality, we can assume, that $Q \subset S_k$.

The representations $\rho_i, i = 1, \ldots, k$ are defined on the group $A$. Lemma 2.3 gives us a possibility to extend domain of these representations to $C$. Let $V_i$ be subspaces of $\mathbb{Q}^n$ corresponding to representations $\rho_i$, $i = 1, \ldots, k$. In fact, since $\rho_{|C}$ is in block diagonal form, we have

$$\forall_{1\leq i\leq k}\ V_i = \underbrace{\Theta \oplus \ldots \oplus \Theta}_{i-1} \oplus \mathbb{Q}^{n/k} \oplus \Theta \oplus \ldots \oplus \Theta \subset \mathbb{Q}^n,$$

where $\Theta$ is considered as a zero-dimensional subspace (zero vector) of $\mathbb{Q}^{n/k}$. Moreover, every element of the group $\rho(Q) = P$ permutes elements of the set

$$\{V_1, \ldots, V_k\}.$$

We want to prove that this action is transitive.

**Lemma 2.5.** *$Q \subset S_k$ is a transitive permutation group.*

*Proof.* If we assume that $Q$ is not transitive, then

$$\exists_{1\leq j\leq k}\forall_{i\neq j}\forall_{\tau\in Q}\ \tau(i) \neq j.$$

Let

$$\hat{V}_j = \bigoplus_{\substack{i=1\\ i\neq j}}^{k} V_i$$

and $c\tau$, where $c \in C, \tau \in Q$, be any element of $G$. Then

$$\rho(c\tau)(\hat{V}_j) = \rho(c)\rho(\tau)(\hat{V}_j) = \rho(c)\left(\bigoplus_{\substack{i=1\\ i\neq j}}^{k} V_{\tau(i)}\right) = \rho(c)(\hat{V}_j) = \hat{V}_j.$$

Thus $\hat{V}_j \subsetneq \mathbb{Q}^n$ is an invariant subspace of $\rho$ and hence $\rho$ is reducible (over $\mathbb{Q}$). This contradiction proves the lemma. $\square$

The following lemma helps us to understand the structure of the group $G$.

**Lemma 2.6.** *Representations* $\rho_1, \ldots, \rho_k \colon C \to \mathrm{GL}(n, \mathbb{Q})$ *are absolutely irreducible.*

*Proof.* Let $\phi\colon C \to \mathrm{GL}(d, \mathbb{C})$ be a $\mathbb{C}$-irreducible subrepresentation of $\rho_{|C}$. By Clifford's theorem, for the group $C \lhd G$ the representation $\rho_{|C}$ is a sum of conjugates of $\phi$, i.e.

$$\rho_{|C} = \bigoplus_{s=1}^{m} \phi^{(g_s)},$$

where $g_s \in G, s = 1, \ldots, m$ and $g_1 = 1$. For every $1 \leq s \leq m$, $\phi^{(g_s)}$ is a complex subrepresentation of some $\rho_i, i = 1, \ldots, k$. Counting dimensions, we can see, that for every $1 \leq i \leq k$

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j},$$

where

$$\forall_{1 \leq j \leq m/k}\ \rho_{i,j} \in \{\phi^{(g_s)} \,|\, 1 \leq s \leq m\}.$$

Let $V_{i,j} \subset V_i$ be an invariant space under the action of $\rho_{i,j}$, for $1 \leq i \leq k, 1 \leq j \leq m/k$. Taking a suitable basis for $V_i$, $1 \leq i \leq k$, we can assume, that the decopmosition

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j}$$

is given in a block diagonal form:

$$\forall_{1 \leq j \leq m/k}\ V_{i,j} = \underbrace{\Theta \oplus \ldots \oplus \Theta}_{j-1} \oplus \mathbb{C}^{n/m} \oplus \Theta \oplus \ldots \oplus \Theta \subset V_i,$$

where $\Theta$ is a zero-dimensional subspace (zero vector) of $\mathbb{C}^{n/m}$. Note that the images of $\rho_{i|A}, i = 1 \ldots, k$, remain the same in this new basis. Hence the description of the representatives of the normalizer given in the subsection 2.2, remains the same for the group $\mathrm{GL}(n, \mathbb{C})$ and we can assume, that $\rho(Q) = P$.

If the representations $\rho_i, i = 1, \ldots, k$, are $\mathbb{C}$-reducible, then $m > k$. Let

$$W = \bigoplus_{i=1}^{k} V_{i,1}$$

and $c\tau, c \in C, \tau \in Q$, be any element of $G$. We get

$$\rho(c\tau)(W) = \rho(c)\rho(\tau)(W) = \rho(c)\left(\bigoplus_{i=1}^{k} V_{\tau(i),1}\right) = \rho(c)(W) = W.$$

Hence $W \subsetneq \mathbb{C}^n$ is an invariant subspace of $\rho$ and thus $\rho$ cannot be absolutely irreducible. Contradition. □

## 2.4 Abelian normal subgroups

Without lose of generality, we can assume, that $A$ is maximal abelian subgroup of $G$, i.e. if $A' \lhd G$ is abelian and $A \subset A'$, then $A = A'$. We will show, that $A$ is unique in $G$ and hence – characteristic.

**Lemma 2.7.** *$A$ is unique in $C$.*

*Proof.* Let $A' \lhd G$ be an abelian group, such that $A' \subset C$. Since all elements of $A$ commute with all elements of $C$, they commute with all elements of $A'$. Hence $AA'$ is normal abelian subgroup of $G$. Since $A$ is maximal, we have

$$AA' = A \Rightarrow A' \subset A.$$

□

If we can prove, that $A \subset C$, then $A$ is going to be unique in $G$. Recall, that by (2.7) we have a short exact sequence

$$1 \longrightarrow C \longrightarrow G \xrightarrow{p} Q \longrightarrow 1.$$

Assuming $A \not\subset C$, we get

$$1 \neq p(A) \lhd Q.$$

We prove that it is impossible.

**Lemma 2.8.** *Let $Q \subset S_k$ be a transitive permutation group and $k \in \mathbb{N}$ be an odd natural number. Then $Q$ does not contain nontrivial normal elementary abelian 2-groups.*

*Proof.* Let us denote by $N(\tau), \tau \in S_k$, a set

$$N(\tau) := \{1 \leq i \leq k \mid \tau(i) \neq i\}.$$

Assume, that $H \lhd Q$ is a normal nontrivial elementary abelian 2-group. Let $\tau$ be any element of $Q$. Without lose of generality we may assume $1 \in N(\tau)$. Since $Q$ is transitive, we have

$$\forall_{1 \leq i \leq k} \exists_{\sigma_i \in Q} \sigma_i(1) = i.$$

Moreover

$$\forall_{1\leq i\leq k} N_i := N(\sigma_i\tau\sigma_i^{-1}) = \sigma_i(N(\tau))$$

and hence

$$\bigcup_{i=1}^{k} N_i = \{1,\ldots,k\}.$$

Let $\mathcal{I}$ be any element of the set

$$\left\{ \mathcal{K} \subset \{1,\ldots,k\} \mid \left| \bigcup_{i\in\mathcal{K}} N_i \right| \text{ is odd} \right\}$$

with a minimum number of elements. Since $\tau$ and all of its conjugates has order 2, $\mathcal{I}$ has at least two elements. Let $s \in \mathcal{I}$. Since $H$ is normal in $Q$, we have

$$\forall_{i\in\mathcal{I}} \sigma_i\tau\sigma_i^{-1} \in H.$$

By the minimality of $\mathcal{I}$, the set

$$N^{(s)} := \bigcup_{i\in\mathcal{I}\setminus\{s\}} N_i$$

contains even number of elements. Moreover, the same applies to the set $N_s = N(\sigma_s\tau\sigma_s^{-1})$. Hence, the intersection

$$N^{(s)} \cap N_s$$

has an odd number of elements. Recall, that $\sigma_i\tau\sigma_i^{-1}$, for $i \in \mathcal{I}$, as elements of order 2, are products of disjoint transpositions. By the above, there exist $t \in \mathcal{I} \setminus \{s\}$ and $a, b, c \in \{1,\ldots,k\}$ such that

$$a \in N_t \setminus N_s, b \in N_t \cap N_s, c \in N_s \setminus N_t$$

and $(a, b)$ and $(b, c)$ are transpositions in $\sigma_t\tau\sigma_t^{-1}$ and $\sigma_s\tau\sigma_s^{-1}$, respectively. But then

$$\sigma_t\tau\sigma_t^{-1} \cdot \sigma_s\tau\sigma_s^{-1}$$

is an element of order greater than 2 in the elementary abelian 2-group $H$. Contradiction. □

We have just proved.

**Proposition 2.9.** *The maximal, normal elementary abelian subgroup $A \lhd G$ is unique maximal in $G$ and hence it is a characteristic subgroup.*

**Corollary 2.10.**

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(G)) \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)).$$

## 2.5 The proof

Let us first restate the theorem.

**Theorem A** *Let $G$ be a finite group with a non-trivial normal abelian subgroup $A$ and let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose $n$ is odd. Then for every $D \in N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue* 1.

*Proof.* Note first, that eigenvalues of matrices and their products does not depend on their conjugacy class. Hence, we can change the basis of $\rho$, with conjugating the group $N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$ by appropriate invertible rational matrix simultaneously, and prove the theorem with these new forms of $\rho$ and $N = N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$. Note that, by $\mathbb{R}$-irreducibility of $\rho$, $N$ is a finite group (see [8, pages 587-588]).

From above, we can assume, that $\rho(A)$ is a group of diagonal matrices. Using Corrolary 2.10, Proposition 2.4 and a fact, that

$$N \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(G)),$$

we get

$$N \subset C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \cdot P.$$

Recall, that

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \bigoplus_{i=1}^{k} \mathrm{GL}(n/k,\mathbb{Q})$$

and elements of $P$ are "block permutation matrices" (see Lemma 2.3 and (2.6) respectively).

Let $D \in N$, then $D$ has the form

$$D = P_\sigma \mathrm{diag}(c_1,\ldots,c_k),$$

where $\sigma \in S_k$ and $c_i \in \mathrm{GL}(n/k,\mathbb{Q})$, for $i = 1,\ldots,k$. Recall, that $G = CQ$, where $Q \subset S_k$ is a transitive permutation group (see Lemma 2.5). Hence there exists $\tau \in Q$, such that

$$\tau(1) = \sigma^{-1}(1).$$

We get

$$P_\tau P_\sigma \mathrm{diag}(c_1,\ldots,c_k) = P_{\sigma\tau}\mathrm{diag}(c_1,\ldots,c_k) = \mathrm{diag}(c_1, X),$$

where $X$ is a matrix of rows of $\mathrm{diag}(c_2,\ldots,c_k)$ permuted by $\sigma\tau$. Since $c_1 \in \mathrm{GL}(n/k,\mathbb{Q})$ has an odd degree, it must have real eigenvalue and since $N$ is

of a finite order, this eigenvalue is $\pm 1$. If the eigenvalue is 1, then we take $g = \tau$ and the theorem is proved. Otherwise, by the Clifford's theorem and the faithfulness of $\rho$, we can take such $a \in A$, that $\rho_1(a) = -\mathrm{I}$. Then $\rho_1(a)c_1$ has an eigenvalue 1 and hence, taking $g = a\tau$, the element

$$\begin{aligned}\rho(g)D = \rho(a\tau)D = \rho(a)\rho(\tau)D = \rho(a)P_\tau P_\sigma \mathrm{diag}(c_1, \ldots, c_k) = \\ = (\rho_1 \oplus \ldots \oplus \rho_k)(a) \cdot \mathrm{diag}(c_1, X) = \\ = \mathrm{diag}(\rho_1(a)c_1, (\rho_2 \oplus \ldots \oplus \rho_k)(a)X)\end{aligned}$$

has an eigenvalue equal to 1 also. This finishes the proof. □

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